

A simple online competitive adaptation of Lempel-Ziv compression with efficient random access support

Akashnil Dutta* Reut Levi† Dana Ron‡ Ronitt Rubinfeld§

Abstract

We present a simple adaptation of the Lempel Ziv 78' (LZ78) compression scheme (*IEEE Transactions on Information Theory*, 1978) that supports efficient random access to the input string. Namely, given query access to the compressed string, it is possible to efficiently recover any symbol of the input string. The compression algorithm is given as input a parameter $\epsilon > 0$, and with very high probability increases the length of the compressed string by at most a factor of $(1 + \epsilon)$. The access time is $O(\log n + 1/\epsilon^2)$ in expectation, and $O(\log n/\epsilon^2)$ with high probability. The scheme relies on sparse transitive-closure spanners. Any (consecutive) substring of the input string can be retrieved at an additional additive cost in the running time of the length of the substring. We also formally establish the necessity of modifying LZ78 so as to allow efficient random access. Specifically, we construct a family of strings for which $\Omega(n/\log n)$ queries to the LZ78-compressed string are required in order to recover a single symbol in the input string. The main benefit of the proposed scheme is that it preserves the online nature and simplicity of LZ78, and that for *every* input string, the length of the compressed string is only a small factor larger than that obtained by running LZ78.

1 Introduction

As the sizes of our data sets are skyrocketing it is become important to allow a user to access any desired *portion* of the original data without decompressing the entire dataset. This problem has been receiving quite a bit of recent attention (see e.g. [14, 2, 7, 12, 4, 8, 3]). Compression and decompression schemes that allow fast random-access decompression support have been proposed with the aim of achieving similar compression rates to the known and widely used compression schemes, such as arithmetic coding [15], LZ78 [16], LZ77 [13] and Huffman coding [11].

*CSAIL, MIT, Cambridge MA 02139. E-mail: akashnil@mit.edu.

†School of Computer Science, Tel Aviv University. E-mail: reuti.levi@gmail.com.

‡School of Electrical Engineering, Tel Aviv University. E-mail: danar@eng.tau.ac.il.

§CSAIL, MIT, Cambridge MA 02139 and the Blavatnik School of Computer Science, Tel Aviv University. E-mail: ronitt@csail.mit.edu.

In this work, we focus on adapting the widely used LZ78 compression scheme so as to allow fast random access support. Namely, given access to the compressed string and a location ℓ in the original uncompressed string, we would like to be able to efficiently recover the ℓ -th symbol in the uncompressed string. More generally, the goal is to efficiently recover a substring starting at location ℓ_1 and ending at location ℓ_2 in the uncompressed string. Previously, Lempel Ziv-based schemes were designed to support fast random access, in particular, based on LZ78 [14], LZ77 [12] and as a special case of grammar-based compression [2].

The first basic question that one may ask is whether there is any need at all to modify the LZ78 scheme in order to support fast random access. We formalize the intuition that this is indeed necessary and show that without any modifications every (possibly randomized) algorithm will need time linear in the length of the LZ78-compressed string to recover a single symbol of the uncompressed string.

Having established that some modification is necessary, the next question is how do we evaluate the compression performance of a compression scheme that is a modification of LZ78 and supports efficient random access. As different strings have very different compressibility properties according to LZ78, in order to compare the quality of a new scheme to LZ78, we consider a competitive analysis framework. In this framework, we require that for every input string, the length of the compressed string is a most multiplicative factor of α larger than the length of the LZ78-compressed string, where $\alpha > 1$ is a small constant. For a randomized compression algorithm this should hold with high probability (that is, probability $1 - 1/\text{poly}(n)$ where n is the length of the input string). If this bound holds (for all strings) then we say that the scheme is α -competitive with LZ78.

One additional feature of interest is whether the modified compression algorithm preserves the online nature of LZ78. The LZ78 compression algorithm works by outputting a sequence of *codewords*, where each codeword encodes a (consecutive) substring of the input string, referred to as a *phrase*. LZ78 is online in the sense that if the compression algorithm is stopped at any point, then we can recover all phrases encoded by the codewords output until that point. Our scheme preserves this property of LZ78 and furthermore, supports online random access. Namely, at each point in the execution of the compression algorithm we can efficiently recover any symbol (substring) of the input string that has already been encoded. A motivating example to keep in mind is of a powerful server that receives a stream of data over a long period of time. All through this period of time the server sends the compressed data to clients which can, in the meantime, retrieve portions of the data efficiently. This scenario fits cases where the data is growing incrementally, as in log files or user-generated content.

1.1 Our Results

We first provide a deterministic compression algorithm which is 3-competitive with LZ78 (as defined above), and a matching random access algorithm which runs in time $O(\log n)$, where n is the length of the input string. This algorithm retrieves any requested single symbol of the uncompressed string. By slightly adapting this algorithm it is possible to retrieve a substring of length s in time $O(\log n) + s$.

Thereafter, we provide a randomized compression algorithm which for any chosen ep-

silon is $(1 + \epsilon)$ -competitive with LZ78. The expected running time of the matching random access algorithm is $O(\log n + 1/\epsilon^2)$, and with high probability is bounded by ¹ $O(\log n/\epsilon^2)$. The probability is taken over the random coins of the randomized compression algorithm. As before, a substring can be recovered in time that is the sum of the (single symbol) random access time and the length of the string. Similarly to LZ78, the scheme works in an online manner in the sense described above. The scheme is fairly simple and does not require any sophisticated data structures. For the sake of simplicity we describe them for the case in which the alphabet of the input string is $\{0, 1\}$, but they can easily be extended to work for any alphabet Σ .

As noted previously, we also give a lower bound that is linear in the length of the compressed string for any random access algorithm that works with (unmodified) LZ78 compressed strings.

Experimental Results. We provide experimental results which demonstrate that our scheme is competitive and that random access is extremely efficient in practice. An implementation of our randomized scheme is available online [5].

1.2 Techniques

The LZ78 compression algorithm outputs a sequence of codewords, each encoding a phrase (substring) of the input string. Each phrase is the concatenation of a previous phrase and one new symbol. The codewords are constructed sequentially, where each codeword consists of an index i of a previously encoded phrase (the longest phrase that matches a prefix of the yet uncompressed part of the input string), and one new symbol. Thus the codewords (phrases they encode) can be seen as forming a directed tree, which is a trie, with an edge pointing from each child to its parent. Hence, if a node v corresponds to a phrase s_1, \dots, s_t , then for each $1 \leq j \leq t$, there is an ancestor node of v that corresponds to the prefix s_1, \dots, s_j , and is encoded by the codeword (i, s_j) (for some i), so that s_j can be “revealed” by obtaining this codeword.

In order to support random access, we want to be able to perform two tasks. The first task is to identify, for any given index ℓ , what is the codeword that encodes the phrase to which the ℓ -th symbol of the input string belongs. We refer to this codeword as the “target codeword”. Let p denote starting position of the corresponding phrase (in the input string), then the second task is to navigate (quickly) up the tree (from the node corresponding to the target codeword) and reach the ancestor node/codeword at depth $\ell - p + 1$ in the tree. This codeword reveals the symbol we are looking for. In order to be able to perform these two tasks efficiently, we modify the LZ78 codewords. To support the first task we add information concerning the position of phrases in the input (uncompressed) string. To support the second task we add additional pointers to ancestor nodes in the tree, that is, indices of encoded phrases that correspond to such nodes. Thus we (virtually) construct a

¹This bound can be improved to $O((\log n/\epsilon + 1/\epsilon^2) \log(\log n/\epsilon))$, but this improvement comes at a cost of making the algorithm somewhat more complicated, and hence we have chosen only to sketch this improvement (see Subsection B.2).

(very sparse) Transitive Closure (TC) spanner [1] on the tree. The spanner allow to navigate quickly between pairs of codes.

When preprocessing is allowed, both tasks can be achieved more efficiently using auxiliary data structures. Specifically, the first task can be achieved using rank and select queries in time complexity $O(1)$ (see e.g. [10]) and the second task can be achieved in time complexity $O(\log \log n)$ via level-ancestor queries on the trie (see e.g. [6]). However, these solutions are not adaptable, at least not in a straightforward way, to the online setting and furthermore the resulting scheme is not $(1 + \epsilon)$ -competitive with LZ78 for every ϵ .

In the deterministic scheme, which is 3-competitive with LZ78, we include the additional information (of the position and one additional pointer) in every codeword, thus making it relatively easy to perform both tasks in time $O(\log n)$. In order to obtain the scheme that is $(1 + \epsilon)$ -competitive with LZ78 we include the additional information only in an $O(\epsilon)$ -fraction of the codewords, and the performance of the tasks becomes more challenging. Nonetheless, the dependence of the running time on n remains logarithmic (and the dependence on $1/\epsilon$ is polynomial).

The codewords which include additional information are chosen randomly in order to spread them out evenly in the trie. It is fairly easy to obtain similar results if the structure of the trie is known in advance, however, in an online setting, the straightforward deterministic approach can blow up the size of the output by a large factor.

1.3 Related Work

Sadakane and Grossi [14] give a compression scheme that supports the retrieval of any s -long consecutive substring of an input string S of length n over alphabet Σ in $O(1 + s/(\log_{|\Sigma|} n))$ time. In particular, for a single symbol in the input string the running time is $O(1)$. The number of bits in the compressed string is upper bounded by $nH_k(S) + O\left(\frac{n}{\log_{|\Sigma|} n} (k \log |\Sigma| + \log \log n)\right)$, where $H_k(S)$ is the k -th order empirical entropy of S . Since their compression algorithm builds on LZ78, the bound on the length of the compressed string for any given input string can actually be expressed as the sum of the length of the LZ78 compressed string plus $\Theta(n \log \log n / \log n)$ bits for supporting rank and select operations in constant time². They build on the LZ78 scheme in the sense that they store suits of data structures that encode the structure of the LZ78 trie and support fast random access. Hence, for input strings that are compressed by LZ78 to a number of bits that is at least on the order of $n \log \log n / \log n$, their result is essentially the best possible as compared to LZ78. However, their scheme is not in general competitive (as defined above) with LZ78 because of its performance on highly compressible strings. We also note that their compression algorithm does not work in an online fashion, but rather constructs all the supporting data structures given the complete LZ78 trie.

Two alternative schemes which give the same space and time bounds as in [14] were provided by González and Navarro [9] and Ferragina and Venturini [7], respectively. They are simpler, where the first uses an arithmetic encoder and the second does not use any compressor. (They also differ in terms of whether k has to be fixed in advance.) By the above

²The $\Theta(n \log \log n / \log n)$ space requirement can be decreased if one is willing to spend more than constant time.

discussion the performance of these schemes is not in general competitive with LZ78.

Kreft and Navarro [12] provide a variant of LZ77 that supports retrieval of any s -long consecutive substring of S in $O(s)$ time. They show that in practice their scheme achieves close results to LZ77 (in terms of the compression ratio). However, the usage of a data structure that supports the rank and select operations requires $\Omega(n \log \log n / \log n)$ bits.

The Lempel-Ziv compression family belongs to a wider family of schemes called grammar-based compression schemes. In these schemes the input string is represented by a context-free grammar (CFG), which is unambiguous, namely, it generates a unique string. Billie et al. [2] show how to transform any grammar-based compression scheme so as to support random access in $O(\log n)$ time. The transformation increases the compressed representation by a multiplicative factor (larger than 1).

2 Preliminaries

The LZ78 compression scheme. Before we describe our adaptation of the LZ78 scheme [16], we describe the latter in detail. The LZ78 compression algorithm receives an input string $x \in \Sigma^n$ over alphabet Σ and returns a list, $C^x = C_{\text{LZ}}^x$, of codewords of the form (i, b) , where $i \in \mathbb{N}$ and $b \in \Sigma$. Henceforth, unless specified otherwise, $\Sigma = \{0, 1\}$. Each codeword (i, b) encodes a phrase, namely a substring of x , which is the concatenation of the i -th phrase (encoded by $C^x[i]$) and b , where we define the 0-th phrase to be the empty string. The first codeword is always of the form $(0, x[1])$, indicating that the first phrase consists of a single symbol $x[1]$. The compression algorithm continues scanning the input string x and partitioning it into phrases. When determining the j -th phrase, if the algorithm has already scanned $x[1, \dots, k]$, then the algorithm finds the longest prefix $x[k+1, \dots, n-1]$ that is the same as a phrase with index $i < j$. If this prefix is $x[k+1, \dots, t]$, then the algorithm outputs the codeword (i, x_{t+1}) (if the prefix is empty, then $i = 0$).

An efficient (linear in n) LZ78 compression algorithm can be implemented by maintaining an auxiliary trie (as illustrated in Figure 2, Section C). The trie structure is implicit in the output of the LZ78 algorithm. Namely, for an input string $x \in \{0, 1\}^n$, the trie $T^x = (V^x, E^x)$ is defined as follows. For each codeword $C^x[i]$, $1 \leq i \leq m$ there is a node v_i in V^x , and there is also a node v_0 corresponding to the root of the tree. If $C^x[j] = (i, b)$, then there is an edge between v_j and v_i (so that v_i is the parent of v_j). Given the correspondence between codewords and nodes in the trie, we shall sometimes refer to them interchangeably.

In the course of the compression process, when constructing the j -th codeword (after scanning $x[1, \dots, k]$) the compression algorithm can find the longest prefix of $x[k+1, \dots, n-1]$ that matches an existing phrase i simply by walking down the trie. Once the longest match is found (the deepest node is reached), a new node is added to the trie. Thus the trie structure may be an actual data structure used in the compression process, but it is also implicit in the compressed string (where we think of a codeword $C^x[j] = (i, b)$ as having a *pointer* to its *parent* $C^x[i]$). Decompression can also be implemented in linear time by iteratively recovering the phrases that correspond to the codewords and essentially rebuilding the trie (either explicitly or implicitly). In what follows, we refer to i as the *index* of $C^x[i]$ and to $x[j]$ as the bit at *position* j .

Competitive schemes with random access support. We aim to provide a scheme, \mathcal{A} , which compresses every input string almost as well as LZ78 and supports efficient local decompression. Namely, given access to a string that is the output of \mathcal{A} on input x and $1 \leq \ell_1 \leq \ell_2 \leq n$, the local decompression algorithm outputs $x[\ell_1, \dots, \ell_2]$ efficiently. In particular, it does so without decompressing the entire string. We first describe our scheme for the case where $\ell_1 = \ell_2$, which we refer to as *random access*, and later explain how to extend the scheme for $\ell_1 < \ell_2$. The quality of the compression is measured with respect to LZ78, formally, we require the scheme to be *competitive* with LZ78 as defined next. We note that here and in all that follows, when we say “with high probability” we mean with probability at least $1 - 1/\text{poly}(n)$.

Definition 1 (Competitive schemes) *Given a pair of deterministic compression algorithms $\mathcal{A} : \{0, 1\}^* \rightarrow \{0, 1\}^*$ and $\mathcal{B} : \{0, 1\}^* \rightarrow \{0, 1\}^*$, we say that algorithm \mathcal{B} is α -competitive with \mathcal{A} if for every input string $x \in \{0, 1\}^*$, we have $|C_{\mathcal{B}}^x| \leq \alpha |C_{\mathcal{A}}^x|$, where $C_{\mathcal{B}}^x$ and $C_{\mathcal{A}}^x$ are the compressed strings output by \mathcal{A} and \mathcal{B} , respectively, on input x . For a randomized algorithm \mathcal{B} , the requirement is that $|C_{\mathcal{B}}^x| \leq \alpha |C_{\mathcal{A}}^x|$ with high probability over the random coins of \mathcal{B} .*

Word RAM model. We consider the RAM model with word size $\log n + 1$, where n is the length of the input string. We note that it suffices to have an upper bound on this value in order to have a bound on the number of bits for representing any index of a phrase. A codeword of LZ78 is one word, i.e., i and b appear consecutively where i is represented by $\log n$ bits. For the sake of clarity of the presentation, we write it as (i, b) . Our algorithms (which supports random access) use words of size $\log n + 1$ as well. If one wants to consider variants of LZ78 that apply bit optimization and/or work when an upper bound on the length of the input string is not known in advance, then our algorithms need to be modified accordingly so as to remain competitive (with the same competitive ratio).

We wish to point out that if we take the word size to be $\log m + 1$ (instead of $\log n + 1$), where m is the number of phrases in the compressed string, then our results remain effectively the same. Specifically, in the worst case the blow up in the deterministic scheme is of factor of 4 (instead of 3) and in the randomized scheme is of factor $(1 + 2\epsilon)$ (instead of $(1 + \epsilon)$).

3 A Deterministic Scheme

In this section we describe a simple deterministic compression scheme which is based on the LZ78 scheme.

In the deterministic compression scheme, to each codeword we add a pair of additional entries. The first additional entry is the starting position of the encoded phrase in the uncompressed string. On an input $x \in \{0, 1\}^n$ and $1 \leq \ell \leq n$, this allows the algorithm to efficiently find the codeword encoding the phrase that contains the ℓ -th bit by performing a binary search on the position entries. The second entry we add is an extra pointer (we shall use the terms “pointer” and “index” interchangeably). Namely, while in LZ78 each codeword indicates the index of the former codeword, i.e., the direct parent in the trie, (see

Section 2), we add another index, to an ancestor node/codeword (which is not the direct parent). In order to allow efficient random access, our goal is to guarantee that for every pair of connected nodes, u, v there is a short path connecting u and v . Namely, if we let $d_G(u, v)$ denote the length of the shortest path from u to v in a directed graph G , then the requirement is that for u, v such that $d_G(u, v) < \infty$ it holds that $d_G(u, v)$ is small. Before we describe how to achieve this property on (a super-graph of) the constructed trie we describe how to guarantee the property on a simple directed path. Formally we are interested in constructing a Transitive-Closure (TC) spanner, defined as follows:

Definition 2 (TC-spanner [1]) *Given a directed graph $G = (V, E)$ and an integer $k \geq 1$, a k -transitive-closure-spanner (k -TC-spanner) of G is a directed graph $H = (V, E_H)$ with the following properties:*

1. E_H is a subset of the edges in the transitive closure³ of G .
2. For all vertices $u, v \in V$, if $d_G(u, v) < \infty$, then $d_H(u, v) \leq k$.

3.1 TC Spanners for Paths and Trees

Let $\mathcal{L}_n = (V, E)$ denote the directed line (path) over n nodes (where edges are directed “backward”). Namely, $V = \{0, \dots, n-1\}$ and $E = \{(i, i-1) : 1 \leq i \leq n-1\}$. Let $f_n(i) \stackrel{\text{def}}{=} i \bmod \lfloor \log n \rfloor$ and let $E' = \{(i, \max\{i - 2^{f_n(i)} \cdot \lfloor \log n \rfloor, 0\}) : 1 \leq i \leq n-1\}$. Observe that each node $1 \leq i \leq n-1$ has exactly one outgoing edge in E' (in addition to the single outgoing edge in E). Define $\mathcal{H}_n = (V, E \cup E')$.

Claim 1 \mathcal{H}_n is a $(4 \log n)$ -TC-spanner of \mathcal{L}_n .

Proof: For every $0 \leq r < t \leq n-1$, consider the following algorithm to get from t to r (at each step of the algorithm stop if r is reached):

1. Starting from t and using the edges of E , go to the first node u such that $f_n(u) = \lfloor \log n \rfloor - 1$.
2. From u iteratively proceed by taking the outgoing edge in E' if it does not go beyond r (i.e., if the node reached after taking the edge is not smaller than r), and taking the outgoing edge in E otherwise.

Clearly, when the algorithm terminates, r is reached. Therefore, it remains to show that the length of the path taken by the algorithm is bounded by $4 \log n$. Let $a(i)$ denote the node reached by the algorithm after taking i edges in E starting from u . Therefore, $a(0) = u$ and $f_n(a(i)) = \lfloor \log n \rfloor - 1 - i$ for every $0 \leq i < \lfloor \log n \rfloor$ and $i \leq s$, where s denotes the total number of edges taken in E starting from u . For every pair of nodes $w \geq q$ define $g(w, q) = \lfloor (w - q) / \lfloor \log n \rfloor \rfloor$, i.e., the number of complete blocks between w and q . Thus, $g(a(i), r)$ is monotonically decreasing in i , for $i \leq s$. Consider the bit representation of $g(a(i), r)$. If from node $a(i)$ the algorithm does not take the edge in E' it is implied that

³The transitive closure of a graph $G = (V, E)$ is the graph $H = (V', E')$ where $V' = V$ and $E' = \{(u, v) : d_G(u, v) < \infty\}$.

the j -th bit in $g(a(i), r)$ is 0 for every $j \geq f_n(a(i))$. On the other hand, if from node $a(i)$ the algorithm takes the edge in E' then after taking this edge the $f_n(a(i))$ -th bit turns 0. Therefore by an inductive argument, when the algorithm reaches $a(i)$, $g(a(i), r)$ is 0 for every $j > f_n(a(i))$. Thus, $g(a(\min\{\lfloor \log n \rfloor - 1, s\}), r) = 0$, implying that the total number of edges taken on E' is at most $\log n$. Combined with the fact that the total number of edges taken on E in Step 2 is bounded by $2 \log n$ and the fact that the total number of edges taken on E in Step 1 is bounded by $\log n$, the claim follows. ■

From Claim 1 it follows that for every $m < n$, $V = \{0, \dots, m\}$, $E = \{(i, i-1) : 1 \leq i \leq m-1\}$ and $E' = \{(i, \max\{i - 2^{f_n(i)} \cdot \lfloor \log n \rfloor, 0\})\}$, $(V, E \cup E')$ is a $(4 \log n)$ -TC-spanner of \mathcal{L}_m . This implies a construction of a $(4 \log n)$ -TC-spanner for any tree on n nodes. Specifically, we consider trees where the direction of the edges is from child to parent (as defined implicitly by the codewords of LZ78) and let $d(v)$ denoted the depth of a node v in the tree (where the depth of the root is 0). If in addition to the pointer to the parent, each node, v , points to the ancestor at distance $2^{f_n(d(v))} \cdot \lfloor \log n \rfloor$ (if such a node exists), then for every pair of nodes u, v on a path from a leaf to the root, there is a path of length at most $4 \log n$ connecting u and v .

We note that using k -TC-spanners with $k = o(\log n)$ will not improve the running time of our random access algorithms asymptotically (since they perform an initial stage of a binary search).

3.2 Compression and Random Access Algorithms

As stated at the start of this section, in order to support efficient random access we modify the codewords of LZ78. Recall that in LZ78 the codewords have the form (i, b) , where i is the index of the parent codeword (node in the trie) and b is the additional bit. In the modified scheme, codewords are of the form $W = (p, i, k, b)$, where i and b remain the same, p is the starting position of the encoded phrase in the uncompressed string and k is an index of an ancestor codeword (i.e., encoding a phrase that is a prefix of the phrase encoded by W). As in LZ78, our compression algorithm (whose pseudo-code appears in Algorithm 1, Subsection A.1) maintains a trie \mathcal{T} as a data structure where the nodes of the trie correspond to codewords encoding phrases (see Section 2). Initially, \mathcal{T} consists of a single root node. Thereafter, the input string is scanned and a node is added to the trie for each codeword that the algorithm outputs, giving the ability to efficiently construct the next codewords. The data structure used is standard: for each node the algorithm maintains the index of the phrase that corresponds to it, its depth, and pointers to its children.

Given access to a compressed string, which is a list of codewords $C[1, \dots, m]$, and an index $1 \leq \ell \leq n$, the random access algorithm (whose pseudo-code appears in Algorithm 2, Subsection A.1) first performs a binary search (using the position entries in the codewords) in order to find the codeword, $C[t]$, which encodes the phrase $x[\ell_1, \dots, \ell_2]$ containing the ℓ -th bit of the input string x (i.e., $\ell_1 \leq \ell \leq \ell_2$). The algorithm then reads $O(\log n)$ codewords from the compressed string, using the parent and ancestor pointers in the codewords, in order to go up the trie (implicitly defined by the codewords) to the node at distance $\ell_2 - \ell$ from the node corresponding to $C[t]$. The final node reached corresponds to the codeword, $C[r] = (p_r, i_r, k_r, b_r)$, which encodes the phrase $x[p_r, \dots, \ell - \ell_1 + 1] = x[\ell_1, \dots, \ell]$ and so

the algorithm returns b_r .

The next theorem follows directly from the description of the algorithms and Claim 1.

Theorem 1 *Algorithm 1 (compression algorithm) is 3-competitive with LZ78, and for every input $x \in \{0, 1\}^n$, the running time of Algorithm 2 (random access algorithm) is $O(\log n)$.*

Recovering a substring. We next describe how to recover a consecutive substring $x[\ell_1, \dots, \ell_2]$, given the compressed string $C[1, \dots, m]$. The idea is to recover the substring in reverse order as follows. Find the codeword, $C[k]$ encoding the substring (phrase) $x[t_1, \dots, t_2]$ such that $t_1 \leq \ell_2 \leq t_2$ as in Step 1 of Algorithm 2. Then, as in Step 2 of Algorithm 2 find the codeword, $C[t]$, which encodes $x[t_1, \dots, \ell_2]$. From $C[t]$ recover the rest of the substring ($x[t_1, \dots, \ell_2 - 1]$) by going up the trie. If the root is reached before recovering $\ell_2 - \ell_1 + 1$ bits (i.e., $\ell_1 < t_1$), then continue decoding $C[k - 1]$, $C[k - 2]$, \dots until reaching the encoding of the phrase within which $x[\ell_1]$ resides. The running time is the sum of the running time of a single random access execution, plus the length of the substring.

4 A Randomized Scheme

In this section we present a randomized compression scheme which builds on the deterministic scheme described in Section 3. In what follows we describe the randomized compression algorithm and the random access algorithm. Their detailed pseudo-codes are given in Algorithm 3 (see Subsection A.2) and Algorithm 4 (see Subsection A.1), respectively. Recovering a substring is done in the same manner as described for the deterministic scheme.

We assume that $\epsilon = \Omega(\log n / \sqrt{\log n})$ (or else one might as well compress using LZ78 without any modifications).

The high-level idea of the compression scheme. Recall that the deterministic compression algorithm (Algorithm 1), which was 3-competitive, adds to each LZ78 codeword two additional information entries: the starting position of the corresponding phrase, and an additional index (pointer) for navigating up the trie. The high level idea of the randomized compression algorithm, which is $(1 + \epsilon)$ -competitive, is to “spread” this information more sparsely. That is, rather than maintaining the starting position of every phrase, it maintains the position only for a $\Theta(\epsilon)$ -fraction of the phrases, and similarly only $\Theta(\epsilon)$ -fraction of the nodes in the trie have additional pointers for “long jumps”. While spreading out the position information is done deterministically (by simply adding this information once in every $\Theta(1/\epsilon)$ codewords), the additional pointers are added randomly (and independently). Since the trie structure is not known in advance, this ensures (with high probability) that the number of additional pointer entries is $O(\epsilon)$ times the number of nodes (phrases), as well as ensuring that the additional pointers are fairly evenly distributed in each path in the trie. We leave it as an open question whether there exists a deterministic (online) algorithm

that always achieves such a guarantee ⁴.

Because of the sparsity of the position and extra-pointer entries, finding the exact phrase to which an input bit belongs and navigating up the trie in order to determine this bit, is not as self-evident as it was in the deterministic scheme. In particular, since the position information is added only once every $\Theta(1/\epsilon)$ phrases, a binary search (similar to the one performed by the deterministic algorithm) for a location ℓ in the input string does not uniquely determine the phrase to which the ℓ -th bit belongs. In order to facilitate finding this phrase (among the $O(1/\epsilon)$ potential candidates), the compression algorithm adds one more type of entry to an $O(\epsilon)$ -fraction of the nodes in the trie: their depth (which equals the length of the phrase to which they correspond). This information also aids the navigation up the trie, as will be explained subsequently.

A more detailed description of the compression algorithm. Similarly to the deterministic compression algorithm, the randomized compression algorithm scans the input string and outputs codewords containing information regarding the corresponding phrases (where the phrases are the same as defined by LZ78). However, rather than having just one type of codeword, it has three types:

- A *simple* codeword of the form (i, b) , which is similar to the codeword LZ78 outputs. Namely, i is a pointer to a former codeword (which encodes the previously encountered phrase that is the longest prefix of the current one), and b is a bit. Here, since the length of the codewords is not fixed, the pointer i indicates the starting position of the former codeword in the compressed string rather than its index. We refer to i as the *parent* entry, and to b as the *value* entry.
- A *special* codeword, which encodes additional information regarding the corresponding node in the trie. Specifically, in addition to the entries i and b as in a simple codeword, there are three additional entries. One is the *depth* of the corresponding node, v , in the tree, and the other two are pointers (starting positions in the compressed string) to special codewords that correspond to ancestors of v . We refer to one of these entries as the *special_parent* and the other as the *special_ancestor*. Details of how they are selected are given subsequently.
- A *position* codeword, which contains the starting position of the next encoded phrase in the uncompressed string.

In what follows we use the term *word* (as opposed to *codeword*) to refer to the RAM words of which the codewords are built. Since codewords have different types and lengths (in terms of the number of words they consist of), the compression algorithm adds a special

⁴The simple idea of adding an extra pointer to all the nodes whose depth is divisible by $k = \Theta(1/\epsilon)$, excluding nodes with height smaller than k , will indeed ensure the even distribution on each path. However, since we do not know the height of each node in advance, if we remove this exclusion we might cause the number of additional pointers to be too large, e.g., if the trie is a complete binary tree with height divisible by k , then every leaf gets an additional pointer.

delimiter word before each special codeword and (a different one) before each position codeword.⁵

The algorithm includes a position codeword every c/ϵ words (where c is a fixed constant). More precisely, since such a word might be in the middle of a codeword, the position codeword is actually added right before the start of the next codeword (that is, at most a constant number of words away). As stated above, the position is the starting position of the phrase encoded by the next codeword.

Turning to the special codewords, each codeword that encodes a phrase is selected to be a special codeword independently at random with probability ϵ/c . We refer to the nodes in the trie that correspond to special codewords as *special nodes*. Let u be a special node (where this information is maintained using a Boolean-valued field named ‘special’). In addition to a pointer i to its parent node in the trie, it is given a pointer q to its closest ancestor that is a special node (its *special parent*) and a pointer a to a *special ancestor*. The latter is determined based on the *special depth* of u , that is, the number of special ancestors of u plus 1, similarly to the way it is determined by the deterministic algorithm. Thus, the special nodes are connected among themselves by a TC-spanner (with out-degree 2).

A more detailed description of the random access algorithm. The random access algorithm Algorithm 4, is given access to a string S , which was created by the randomized compression algorithm, Algorithm 3. This string consists of codewords $C[1], \dots, C[m]$ (of varying lengths, so that each $C[j]$ equals $S[r, \dots, r+h]$ for $h \in \{0, 1, 4\}$). Similarly to Algorithm 2 for random access when the string is compressed using the deterministic compression algorithm, Algorithm 4, the algorithm for random access when the string is compressed using the randomized compression algorithm, consists of two stages. Given an index $1 \leq \ell \leq n$, in the first stage the algorithm finds the codeword that encodes the phrase $x[\ell_1, \dots, \ell_2]$ to which the ℓ -th bit of the input string x belongs (so that $\ell_1 \leq \ell \leq \ell_2$). In the second stage it finds the codeword that encodes the phrase $x[\ell_1, \dots, \ell]$ (which appeared earlier in the string), and returns its value entry (i.e., the bit b).

Recall that on input ℓ and $C[1, \dots, m]$, Algorithm 2 (in Step 1) first finds the codeword that encodes the phrase to which the ℓ -th bit of the input string belongs by performing a binary search. This is done using the position entries, where each codeword has such an entry. However, in the output string of the randomized compression scheme it is no longer the case that each codeword has a position entry. Still, the random access algorithm can perform a binary search over the position codewords. Recall that the randomized compression algorithm places these codewords at almost fixed positions in the compressed string (namely, at positions that are at most a constant number of words away from the fixed positions), and these codewords are marked by a delimiter. Hence, the algorithm can find two position codewords, $C[k]$ and $C[q]$, such that $q - \ell = O(1/\epsilon)$ and such that ℓ is between the positions corresponding to these codewords. This implies that the requested bit $x[\ell]$ belongs to one of the phrases associated with the codewords $C[k+1], \dots, C[q-1]$.

In order to find the desired codeword $C[t]$ where $k < t < q$, the algorithm calculates the length of the phrase each of the codewords $C[k+1], \dots, C[q-1]$ encodes. This length

⁵In particular, these can be the all-1 word and the word that is all-1 with the exception of the last bit, which is 0. This is possible because the number of words in the compressed string is $O(n/\log n)$.

equals the depth of codeword (corresponding node) in the trie. If a codeword is a special codeword, then this information is contained in the codeword. Otherwise (the codeword is a simple codeword), the algorithm computes the depth of the corresponding node by going up the trie until it reaches a special node (corresponding to a special codeword). Recall that a walk up the tree can be performed using the basic parent pointers (contained in both simple and special codewords), and that each special codeword is marked by a delimiter, so that it can be easily recognized as special. (For the pseudo-code see Procedure Find-Depth in Subsection A.2.)

Let the phrase encoded by $C[t]$ be $x[\ell_1, \dots, \ell_2]$ (where $\ell_1 \leq \ell \leq \ell_2$). In the second stage, the random access algorithm finds the codeword, $C[r]$, which encodes the phrase $x[\ell_1, \dots, \ell]$ (and returns its value entry, b , which equals $x[\ell]$). This is done in three steps. First the algorithm uses parent pointers to reach the special node, v , which is closest to the node corresponding to $C[t]$. Then the algorithm uses the special_parent pointers and special_ancestor pointers (i.e., TC-spanner edges) to reach the special node, v' , which is closest to the node corresponding to $C[r]$ (and is a descendent of it). This step uses the depth information that is provided in all special nodes in order to avoid “over-shooting” $C[r]$. (Note that the depth of the node corresponding to $C[r]$ is known.) Since the special nodes v and v' are connected by an $O(\log n)$ -TC-spanner, we know (by Claim 1) that there is a path of length $O(\log n)$ from v to v' . While the algorithm does not know what is the depth of v' , it can use the depth of the node corresponding to $C[r]$ instead to decide what edges to take. In the last step, the node corresponding to $C[r]$ is reached by taking (basic) parent pointers from v' .

Theorem 2 *Algorithm 3 is $(1 + \epsilon)$ -competitive with LZ78 and for every input $x \in \{0, 1\}^n$, the expected running time of Algorithm 4 is $O(\log n + 1/\epsilon^2)$. With high probability over the random coins of Algorithm 3 the running time of Algorithm 4 is bounded by $O(\log n/\epsilon^2)$.*

Proof: For an input string $x \in \{0, 1\}^n$, let $w(x)$ be the number of codewords (and hence words) in the LZ78 compression of x , and let $w'(x)$ be the number of words obtained when compressing with Algorithm 3 (so that $w'(x)$ is a random variable). Let $m'_1(x)$ be the number of simple codewords in the compressed string, let $m'_2(x)$ be the number of special codewords, and let $m'_3(x)$ be the number of position codewords. Therefore, $w'(x) = m'_1(x) + 5m'_2(x) + 2m'_3(x)$. By construction, $m'_1(x) + m'_2(x) = w(x)$, and so $w'(x) = w(x) + 4m'_2(x) + 2m'_3(x)$. Also by construction we have that $m'_3(x) = \epsilon w'(x)/40$, so that $w'(x) = \frac{w(x) + 4m'_2(x)}{1 - \epsilon/20}$. Since each phrase is selected to be encoded by a special codeword independently with probability $\epsilon/40$, by a multiplicative Chernoff bound, the probability that more than an $(\epsilon/20)$ -fraction of the phrases will be selected, i.e., $m'_2(x) > (\epsilon/20)w(x)$ is bounded by $\exp(-\Omega(\epsilon w(x))) < \exp(-\Omega(\epsilon \sqrt{n}))$ (since $w(x) \geq \sqrt{n}$). Therefore, with high probability (recall that we may assume that $\epsilon \geq c \log(n)/\sqrt{n}$ for a sufficiently large constant c) we get that $w'(x) \leq \frac{1 + \epsilon/5}{1 - \epsilon/20} \cdot w(x) \leq (1 + \epsilon)w(x)$. Since the analysis of the running time is easier to follow by referring to specific steps in the pseudo-code of the algorithm (see Subsection A.2) we refer the reader to Subsection B.1 for the rest of the proof. ■

5 A Lower Bound for Random Access in LZ78

In what follows we describe a family of strings, $x \in \{0, 1\}^n$, for which random access to x from the LZ78 compressed string, $C^x = C_{\text{LZ}}^x$, requires $\Omega(|C^x|)$ queries, where $|C^x|$ denotes the number of codewords in C^x . We construct the lower bound for strings, x , such that $|C^x| = \Omega(n/\log n)$ (Theorem 3) and afterwards extend (Theorem 4) the construction for general n and m , where n denotes the length of the uncompressed string and m denotes the number of codewords in the corresponding compressed string. Note that m is lower bounded by $\Omega(\sqrt{n})$ and upper bounded by $O(n/\log n)$. Consider the two extreme cases, the case where the trie, T^x , has a topology of a line, for example when $x = \underline{0} \underline{01} \underline{01^2} \dots \underline{01^j}$. In this case $|C^x| = \Omega(\sqrt{n})$; the case where the trie is a complete tree, corresponding for example to the string that is a concatenation of all the strings up to a certain length, ordered by their length. In the latter case, from the fact that T^x is a complete binary tree on $m + 1$ nodes it follows that x is of length $\Theta(m \log m)$, thus $|C^x| = O(n/\log n)$.

The idea behind the construction is as follows. Assume $m = 2^k - 1$ for some $k \in \mathbb{Z}^+$ and consider the string $S = \underline{0} \underline{1} \underline{00} \underline{01} \underline{10} \underline{11} \underline{000} \dots \underline{1^{k-1}}$, namely, the string that contains all strings of length at most $k - 1$ ordered by their length and then by their lexicographical order. Let S^ℓ denote the string that is identical to S except for the ℓ -th order string, s , amongst strings with prefix 01 and length $k - 1$. We modify the prefix of s from 01 to 00 and add an arbitrary bit to the end of s . The key observation is that the encoding of S and S^ℓ differs in a single location, i.e. a single codeword. Moreover, this location is disjoint for different values of ℓ and therefore implies a lower bound of $\Omega(m)$ as formalized in the next theorem.

Theorem 3 *For every $m = 2^k - 2$ where $k \in \mathbb{Z}^+$, there exist $n = \Theta(m \log m)$, an index $0 \leq i \leq n$ and a distribution, \mathcal{D} , over $\{0, 1\}^n \cup \{0, 1\}^{n+1}$ such that*

1. $|C^x| = m$ for every $x \in \mathcal{D}$.
2. Every algorithm \mathcal{A} for which it holds that $\Pr_{x \in \mathcal{D}} [\mathcal{A}(C^x) = x_i] \geq 2/3$ must read $\Omega(2^k)$ codewords from C^x .

Proof: Let $x \circ y$ denote x concatenated to y and $\bigcirc_{i=1}^t s_i$ denote $s_1 \circ s_2 \dots \circ s_t$. Define $S = \bigcirc_{i=1}^{k-1} \left(\bigcirc_{j=1}^{2^i} s(i, j) \right)$ where $s(i, j)$ is the j -th string, according to the lexicographical order, amongst strings of length i over alphabet $\{0, 1\}$. For every $1 \leq \ell \leq q \stackrel{\text{def}}{=} 2^{k-1}/4$ define $S^\ell = \bigcirc_{i=1}^{k-1} \left(\bigcirc_{j=1}^{2^i} s^\ell(i, j) \right)$ where $s^\ell(i, j) = s(k-1, 1) \circ 0$ for $i = k-1$ and $j = q + \ell$ and $s^\ell(i, j) = s(i, j)$ otherwise. Define $C_{i,j}^x \stackrel{\text{def}}{=} C^x[2^i - 1 + j]$. Therefore, $C_{i,j}^S$ corresponds to the j -th node in the i -th level of the T^S , i.e. $C_{i,j}^S = s(i, j)$ (see Figure 3, Section C). Thus $C_{i,j}^S \neq C_{i,j}^{S^\ell}$ for $\langle i, j \rangle = \langle k-1, q + \ell \rangle$ and $C_{i,j}^S = C_{i,j}^{S^\ell}$ otherwise. We define \mathcal{D} to be the distribution of the random variable that takes the value S with probability $1/2$ and the value S^ℓ with probability $1/(2\ell)$ for every $1 \leq \ell \leq q$. We first argue that for some absolute constant $\eta < 0$, for every algorithm, \mathcal{A} , which for an input C^x takes $\eta|C^x|$ queries from C^x , it holds that $\Pr_{R \in \mathcal{D}} [\mathcal{A}(C^S) \neq \mathcal{A}(C^R)] \leq 1/6$. This follows from the combination of the fact that $q = \Omega(|C^S|)$ and the fact that \mathcal{A} must query the compressed

string on the ℓ -th location in order to distinguish S^ℓ from S . To complete the proof we show that there exists $0 \leq i \leq n$ such that $\Pr_{R \in \mathcal{D}} [C_i^S = C_i^R] = 1/2$, namely, show that $C_i^S \neq C_i^{S^\ell}$ for every $1 \leq \ell \leq q$. Since the position of the phrases of length $k - 1$ with prefix 1 is shifted by one in S^ℓ with respect to S we get that the above is true for $\Omega(|C^S|)$ bits. In particular, $C_i^S \neq C_i^{S^\ell}$ holds for every bit, x_i , that is encoded in the second to last position of a phrase of length $k - 1$ with prefix 1 and suffix 01. ■

Theorem 3 can be extended as follows:

Theorem 4 *For every \tilde{m} and \tilde{n} such that $\tilde{m} \log \tilde{m} < \tilde{n} < \tilde{m}^2$ there exist:*

1. $m = \Theta(\tilde{m})$ and $n = \Theta(\tilde{n})$
2. a distribution, \mathcal{D} , over $\{0, 1\}^n \cup \{0, 1\}^{n+1}$
3. an index $0 \leq i \leq n$

such that Conditions 1 and 2 in Theorem 3 hold.

Proof: Set $k = \lceil \log \tilde{m} \rceil$, $t = \lceil \sqrt{\tilde{n}} \rceil$ and let $m = 2^k - 1 + t$. Define $S = \bigcirc_{i=1}^{k-1} \left(\bigcirc_{j=1}^{2^i} (0 \circ s(i, j)) \right) \bigcirc_{i=1}^t 1^i$ and $S^\ell = \bigcirc_{i=1}^{k-1} \left(\bigcirc_{j=1}^{2^i} (0 \circ s^\ell(i, j)) \right) \bigcirc_{i=1}^t 1^i$. Therefore $n = \Theta(k2^k + t^2) = \Theta(\tilde{n})$. The rest of the proof follows the same lines as in the proof of Theorem 3. ■

6 Experimental Results

Our experiments show that on selected example files our scheme is competitive in practice (see Figure 1). Our results are given below in terms of the fraction of special codewords, α , which is directly related to ϵ (see Theorem 2). We ran the scheme with $\alpha = 1/4, 1/8, 1/16$. The data points corresponding to $\alpha = 0$ plot the file size resulting from standard LZ78.

With respect to the random access efficiency, we found that on average the time required for random access is less than 1 millisecond while decompressing the entire file takes around 300 milliseconds.

Acknowledgment. We would like to thank Giuseppe Ottaviano for a helpful discussion and two anonymous referees for their constructive comments.

References

- [1] A. Bhattacharyya, E. Grigorescu, K. Jung, S. Raskhodnikova, and D. P. Woodruff. Transitive-closure spanners. In *Proceedings of SODA*, pages 932–941, 2009.
- [2] P. Bille, G. M. Landau, R. Raman, K. Sdakane, S. R. Satti, and O. Weimann. Random access to grammar-compressed strings. In *Proceedings of SODA*, pages 373–389, 2011.
- [3] V. Chandar, D. Shah, and G. W. Wornell. A locally encodable and decodable compressed data structure. *47th Annual Allerton Conference on Communication, Control, and Computing, 2009*, pages 613–619, 2009.

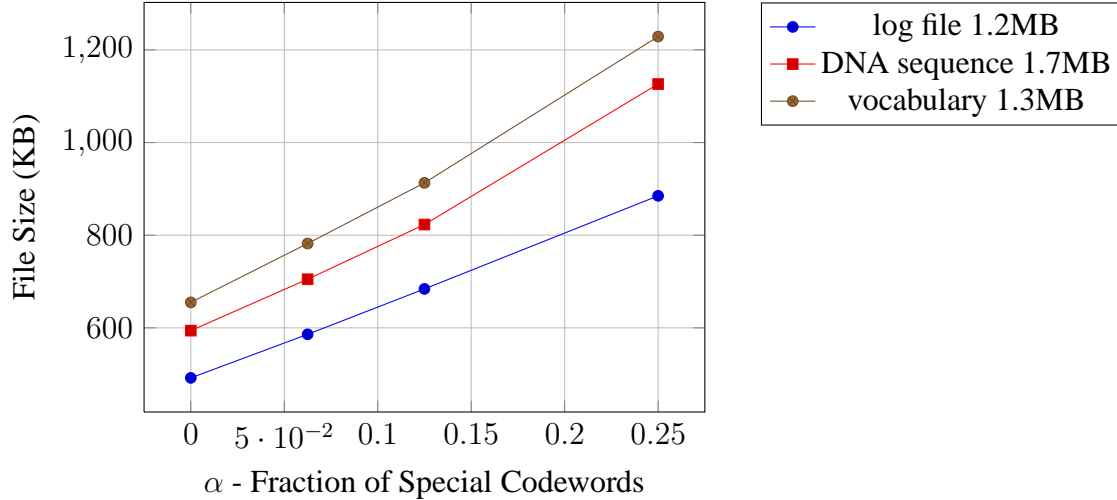


Figure 1: Competitive Ratio

- [4] Y. Dodis, M. Patrascu, and M. Thorup. Changing base without losing space. In *Proceedings of STOC*, pages 593–602, 2010.
- [5] A. Dutta, R. Levi, D. Ron, and R. Rubinfeld. Implementation of the randomized compression scheme. Available from <http://people.csail.mit.edu/ronitt/CODE/LZ78RandomAccess>, 2012.
- [6] A. Farzan and J. I. Munro. A uniform approach towards succinct representation of trees. In Joachim Gudmundsson, editor, *SWAT*, volume 5124 of *Lecture Notes in Computer Science*, pages 173–184. Springer, 2008.
- [7] P. Ferragina and R. Venturini. A simple storage scheme for strings achieving entropy bounds. In *Proceedings of SODA*, pages 690–696, 2007.
- [8] K. Fredriksson and F. Nikitin. Simple random access compression. *Fundamenta Informaticae*, 92(1-2):63–81, 2009.
- [9] R. González and G. Navarro. Statistical encoding of succinct data structures. In *Proceedings of CPM*, pages 295–306, 2006.
- [10] A. Gupta, W. Hon, R. Shah, and J. S. Vitter. Compressed data structures: Dictionaries and data-aware measures. *Theor. Comput. Sci.*, 387(3):313–331, 2007.
- [11] D. A. Huffman. A method for construction of minimum-redundancy codes. *Proceedings IRE*, 40(9):1098–1101, 1952.
- [12] S. Kreft and G. Navarro. LZ77-like compression with fast random access. In *DCC*, pages 239–248, 2010.
- [13] A. Lempel and J. Ziv. A universal algorithm for sequential data compression. *IEEE Transactions on Information Theory*, 23(3):337–343, 1977.
- [14] K. Sadakane and R. Grossi. Squeezing succinct data structures into entropy bounds. In *Proceedings of SODA*, pages 1230–1239, 2006.
- [15] I. H. Witten, R. M. Neal, and J. G. Cleary. Arithmetic coding for data compression. *Communications of the ACM*, 30(6):520–540, 1987.

- [16] J. Ziv and A. Lempel. Compression of individual sequences via variable-rate coding. *IEEE Transactions on Information Theory*, 24(5):530–536, 1978.

A Pseudo-code

A.1 Deterministic Scheme Pseudo-code

Algorithm 1: Deterministic Compression Algorithm

Input: $x \in \{0, 1\}^n$

Initialize \mathcal{T} to a single root node; $p := 1, j := 1$.

While ($p \leq n$)

1. Find a path in \mathcal{T} from the root to the deepest node, v , which corresponds to a prefix of $x[p, \dots, n - 1]$, where 0 corresponds to the left child and 1 corresponds to the right child.
 2. Create a new node u and set $u.\text{index} := j, u.\text{depth} := v.\text{depth} + 1$.
 3. If $x[p + v.\text{depth}] = 0$ set $v.\text{left} := u$ and otherwise set $v.\text{right} := u$.
 4. Let a be the ancestor of u in \mathcal{T} at depth $\max\{u.\text{depth} - 2^{f_n(u.\text{depth})} \cdot \lfloor \log n \rfloor, 0\}$.
 5. Output $(p, v.\text{index}, a.\text{index}, x[p + u.\text{depth}])$.
 6. $p := p + u.\text{depth} + 1$.
 7. $j := j + 1$.
-

Algorithm 2: Random Access Algorithm for Deterministic Scheme

Input: $C[1] = (p_1, i_1, k_1, b_1), \dots, C[m] = (p_m, i_m, k_m, b_m)$, which represents a string compressed by Algorithm 1, and an index $1 \leq \ell \leq n$

1. Perform a binary search on p_1, \dots, p_m and find p_t such that $p_t = \max_{1 \leq i \leq m} \{p_i \leq \ell\}$.
 2. Find the codeword, $C[r] = (p_r, i_r, k_r, b_r)$, which correspond to the ancestor of $C[t] = (p_t, i_t, k_t, b_t)$ at depth $\ell - p_t + 1$ in the trie. This is done as described in the proof of Claim 1 using the pointer information in the codewords/nodes (observe that the depth of $C[t]$ is $p_{t+1} - p_t$).
 3. Output b_r .
-

A.2 Randomized Scheme Pseudo-code

Algorithm 3: Randomized Compression Algorithm

Input: $x \in \{0, 1\}^n, \epsilon$

Initialize T to a root node, $p := 1, j := 1$

While ($p \leq n$)

1. Find a path in \mathcal{T} from the root to a leaf, v , which corresponds to a prefix of $x[p, \dots, n]$, where 0 corresponds to left child and 1 corresponds to right child.
2. Create a new node u and set:
 - $u.\text{index} := j$
 - $u.\text{depth} := v.\text{depth} + 1$
 - $u.\text{special} := 0$
3. If $x[p + v.\text{depth}] = 0$ set $v.\text{left} := u$ and otherwise set $v.\text{right} := u$.
4. $h := j \bmod 40/\epsilon$.
5. Toss a coin c , with success probability $\epsilon/40$.
6. If $c = 1$ output a special codeword as follows:
 - (a) $u.\text{special} := 1$
 - (b) Let \mathcal{P} denote the path in T from u to the root and let q be the first node in \mathcal{P} such that $q.\text{special} = 1$ (if such exists, otherwise $q = 0$).
 - (c) If $q \neq 0$ set $u.\text{special_depth} := q.\text{special_depth} + 1$, otherwise $u.\text{special_depth} := 0$.
 - (d) Let $d := u.\text{special_depth}$. If $d \neq 0$, let a be the special node on \mathcal{P} for which $a.\text{special_depth} = \max \{d - 2^{f_n(d)} \cdot \lfloor \log n \rfloor, 0\}$.
 - (e) $j := j + 4$
 - (f) Output $(\triangle, u.\text{depth}, v.\text{index}, q.\text{index}, a.\text{index}, x[p + u.\text{depth}])$, (\triangle is a delimiter symbol)

Else, output a simple codeword, namely $i, x[p + u.\text{depth}]$.

7. $p := p + u.\text{depth} + 1$.
 8. $j := j + 1$.
 9. If $h > (j \bmod 40/\epsilon)$, output ∇, p (∇ is a delimiter symbol)
-

Algorithm 4: Random Access Algorithm for Randomized Scheme

Input: a string, S , which is the output Algorithm 3, and an index $1 \leq \ell \leq n$. S consists of varying length codewords $C[1], \dots, C[m]$

1. Perform a binary search on the position codewords in S to find a position codeword $C[k]$ such that $C[k].\text{position} \leq \ell$ and $C[q].\text{position} > \ell$ where $C[q]$ is the next position codeword in S .
 2. $p := C[k].\text{position}$
 3. Starting from $C[k + 1]$, scan S and find the codeword, $C[t]$, which encodes the phrase that contains the bit at position ℓ as follows:
 - (a) $t := k + 1$
 - (b) $d := \text{Find-Depth}(C[t])$
 - (c) While $(p + d < \ell)$
 - i. $p := p + d$
 - ii. Read the next codeword, $C[t]$.
 - iii. $d := \text{Find-Depth}(C[t])$
 4. $C[r] := \text{Find-Node-by-Depth}(C[t], \ell - p + 1)$
 5. Output $C[r].\text{value}$
-

Procedure Find-Node-by-Depth(u, d)	Procedure Find-Depth(u)
Input: the source node, u , and the depth of the target node, d	Input: source node u If u is a special node, return $u.depth$. $i := 1$ While($u.parent$ is a simple node)
1. $s := Find-Depth(u) - d$	1. $u := u.parent$
2. While (u is not a special node and $s > 0$) <ul style="list-style-type: none"> (a) $u := u.parent$ (b) $s := s - 1$ 	2. $i := i + 1$
3. $v := u.special_parent$	Return $i + u.depth$
4. While $v.special_ancestor.depth < u.special_ancestor.depth$ <ul style="list-style-type: none"> (a) If ($v.special_parent.depth < d$) then break loop (b) Else, $u := v$ 	
5. While ($u.special_parent.depth \geq d$) <ul style="list-style-type: none"> (a) If ($u.special_ancestor.depth \geq d$) then $u := u.special_ancestor$ (b) Else, $u := u.special_parent$ 	
6. $s := u.depth - d$	
7. While ($s > 0$) <ul style="list-style-type: none"> (a) $u := u.parent$ (b) $s := s - 1$ 	
8. Output u	

B Running Time Analysis and Improvement

B.1 Bounding the Running Time of Algorithm 4

In Step 1, Algorithm 4 performs a binary search, therefore it terminates after at most $\log n$ iterations. In each iteration of the binary search the algorithm scans a constant number of words as guaranteed by Step 9 in Algorithm 3. Hence, the running time of Step 1 is bounded by $O(\log n)$.

In order to analyze the remaining steps in the algorithm, consider any node v in T . Since each node is picked to be special with probability $\epsilon/40$, the expected distance of any node to the closest special node is $O(1/\epsilon)$. Since the choice of special nodes is done independently, the probability that the closest special ancestor is at distance greater than $40c \log n/\epsilon$ is $(1 - \epsilon/40)^{40c \log n/\epsilon} < 1/n^c$. By taking a union bound over all $O(n)$ nodes, with high probability, for every node v the closest special ancestor is at distance $O(\log n/\epsilon)$.

The first implication of the above is that the running time of Procedure Find-Depth is $O(1/\epsilon)$ in expectation, and with high probability every call to Procedure Find-Depth takes time $O(\log n/\epsilon)$. Hence Step 3 in Algorithm 4 takes time $O(1/\epsilon^2)$ in expectation and $O(\log n/\epsilon^2)$ with high probability. It remains to upper bound the running time of Procedure Find-Node-by-Depth (see Subsection A.2), which is called in Step 4 of Algorithm 4.

With high probability, the running time of Steps 1, 2 and 7 in Procedure Find-Node-by-Depth is $O(1/\epsilon)$ in expectation, and $O(\log n/\epsilon)$ with high probability. The running time of Step 4 is $O(\log n)$ by the definition of the TC-spanner over the special nodes. Finally, by the explanation following the description of the algorithm regarding the relation between Step 5 in Procedure Find-Node-by-Depth and the path constructed in the proof of Claim 1, the running time of Step 5 is $O(\log n)$ as well. Summing up all contribution to the running time we get the bounds stated in the lemma.

B.2 Improving the Running Time from $O(\log n/\epsilon^2)$ to $\tilde{O}(\log n/\epsilon + 1/\epsilon^2)$

As can be seen from the proof of Theorem 2, the dominant contribution to the running time of the random access algorithm (Algorithm 4) in the worst case (which holds with high probability) comes from Step 3 of the algorithm. We bounded the running time of this step by $O(\log n/\epsilon^2)$ while the running time of the others steps is bounded by $O(\log n/\epsilon)$. In this step the algorithm computes the length of $O(1/\epsilon)$ phrases by determining the depth in the trie of their corresponding nodes. This is done by walking up the trie until a special node is reached. Since we bounded (with high probability) the distance of every node to the closest special node by $O(\log n/\epsilon)$, we got $O(\log n/\epsilon^2)$. However, by modifying the algorithm and the analysis, we can decrease this bound to $\tilde{O}(\log n/\epsilon + 1/\epsilon^2)$. Since this modification makes the algorithm a bit more complicated, we only sketch it below.

Let v_1, \dots, v_k , where $k = O(1/\epsilon)$ be the nodes whose depth we are interested in finding. Let T' be the tree that contains all these nodes and their ancestors in the trie. Recall that the structure of the trie is determined by the LZ78 parsing rule, which is used by our compression algorithm, and that the randomization of the algorithm is only over the choices of the special nodes. To gain intuition, consider two extreme cases. In one case

T' consists of a long path, at the bottom of which is a complete binary tree, whose nodes are v_1, \dots, v_k . In the other extreme, the least common ancestor of any two nodes v_i and v_j among v_1, \dots, v_k , is very far away from both v_i and v_j . Consider the second case first, and let X_1, \dots, X_k be random variables whose value is determined by the choice of the special nodes in T' , where X_i is the distance from v_i to its closest ancestor that is a special node. In this (second) case X_1, \dots, X_k are almost independent. Assuming they were truly independent, it is not hard to show that with high probability (i.e., $1 - 1/\text{poly}(n)$), not only is each X_i upper bounded by $O(\log n/\epsilon)$, but so is their sum. Such a bound on the sum of the X_i 's directly gives a bound on the running time of Step 3.

In general, these random variables may be very dependent. In particular this is true in the first aforementioned case. However, in this (first) case, even if none of the nodes in the small complete tree are special, and the distance from the root of this tree to the closest special node is $\Theta(\log n/\epsilon)$, we can find the depth of all nodes v_1, \dots, v_k in time $O(\log n/\epsilon)$ (even though the sum of their distances to the closest special node is $O(\log n/\epsilon^2)$). This is true because once we find the depth of one node by walking up to the closest special node, if we maintain the information regarding the nodes passed on the way, we do not have to take the same path up T' more than once. Maintaining this information can be done using standard data structures at a cost of $O(\log(\log n/\epsilon))$ per operation. As for the analysis, suppose we redefine X_i to be the number of steps taken up the trie until either a special node is reached, or another node whose depth was already computed is reached. We are interested in upper bounding $\sum_{i=1}^k X_i$. Since these random variables are not independent, we define a set of i.i.d. random variables, Y_1, \dots, Y_k , where each Y_i is the number of coins flipped until a 'HEADS' is obtained, where each coin has bias ϵ/c . It can be verified that by showing that with high probability $\sum_{i=1}^k Y_i = O(\log n/\epsilon)$ we can get the same bound for $\sum_{i=1}^k X_i$, and such a bound can be obtained by applying a multiplicative Chernoff bound.

C Figures

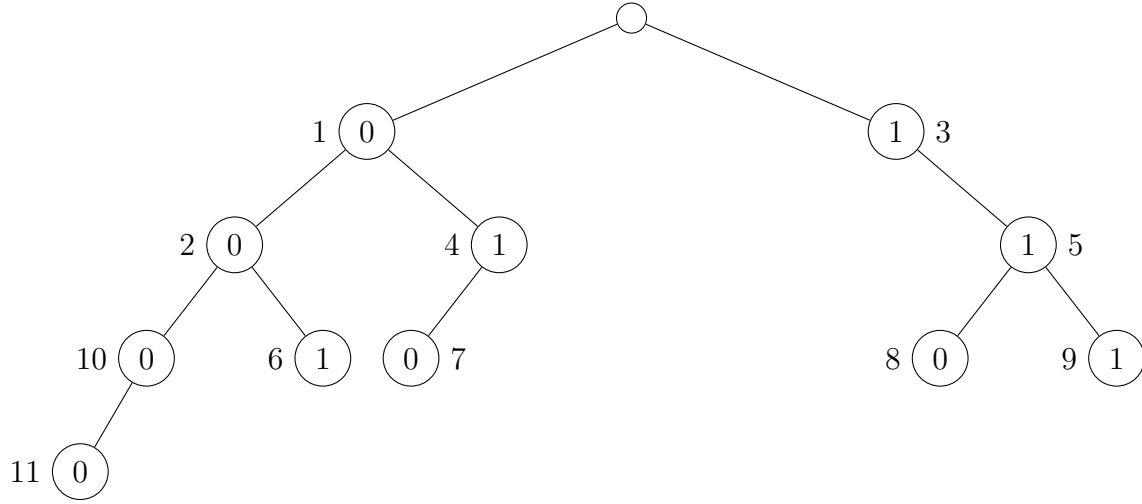


Figure 2: The trie, T^x , implicitly defined by the LZ78 scheme on the string $x = \underline{0} \ \underline{00} \ \underline{1} \ \underline{01} \ \underline{11} \ \underline{001} \ \underline{010} \ \underline{110} \ \underline{111} \ \underline{000} \ \underline{0000}$; On input string x , the LZ78 scheme outputs a list of codewords, $C^x = \{(0, 0), (1, 0), (0, 1), (1, 1), (3, 1), (2, 1), (4, 0), (5, 0), (5, 1), (2, 0), (10, 0)\}$.

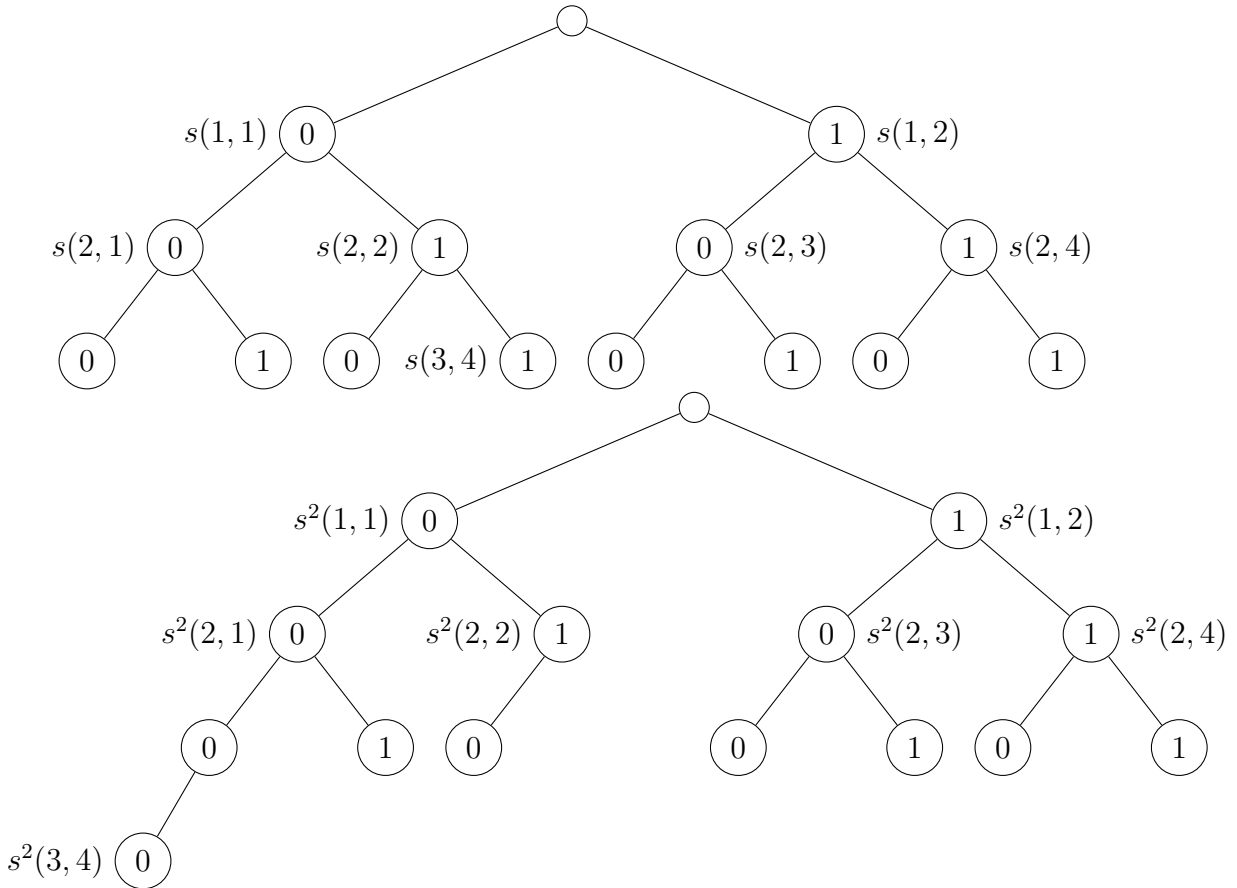


Figure 3: $T(S)$ and $T(S^2)$